Real Zeros and Partitions without singleton blocks

Miklós Bóna

Department of Mathematics
University of Florida
Gainesville FL 32611-8105
USA

February 5, 2008

Abstract

We prove that the generating polynomials of partitions of an n-element set into non-singleton blocks, counted by the number of blocks, have real roots only. We apply this information to find the most likely number of blocks. As another application of the real zeros result, we prove that the number of blocks is normally distributed in such partitions. We present a quick way to prove the corresponding statement for cycles of permutations in which each cycle is longer than a given integer r.

1 Introduction

A *partition* of the set $[n] = \{1, 2, \cdots, n\}$ is a set of blocks disjoint blocks $B_1, B_2, \cdots, B_k$ so that $\cup_{i=1}^{k} B_i = [n]$. The number of partitions of $[n]$ into k blocks is denoted by $S(n, k)$ and is called a *Stirling number of the second kind*.

Similarly, the number of permutations of length n with exactly k cycles is denoted by $c(n, k)$, and is called a *signless Stirling number of the first kind*. See any textbook on Introductory Combinatorics, such as [2] or [3] for the relevant definitions, or basic facts, on Stirling numbers.

The "horizontal" generating functions, or generating polynomials, of Stirling numbers have many interesting properties. Let n be a fixed positive integer. Then it is well-known (see [2] or [3] for instance) that

$$C_n(x) = \sum_{k=1}^{n} c(n, k) x^k = x(x+1) \cdots (x+n-1). \tag{1}$$

1

In particular, the roots of the generating polynomial $C_n(x)$ are all real (indeed, they are the integers $0, -1, -2, \cdots, -(n-1)$).

Similarly, it is known (see [16], page 20, for instance) that for any fixed positive integer n, the roots of the generating polynomial

$$S_n(x) = \sum_{k=1}^{n} S(n,k)x^k$$

are all real, though they are not nearly as easy to describe as those of $C_n(x)$.

Rodney Canfield [8] (in the case of $r = 1$) and Francesco Brenti [5] (in the general case) have generalized (1) as follows. Let $d_r(n,k)$ be the number of permutations of length n that have k cycles, each longer than r. Such permutations are sometimes called r-derangements. Then the generating polynomial

$$d_{n,r}(x) = \sum_{k \geq 1} d_r(n,k)x^k \tag{2}$$

has real roots only. Present author [4] proved that for any given positive integer constant m, there exists a positive number N so that if $n > N$, then one of these roots will be very close to -1, one will be very close to -2, and so on, with one being very close to $-m$, to close the sequence of m roots being very close to consecutive negative integers.

In this paper, we consider the analogue problem for set partitions. Let $D(n,k)$ be the number of partitions of $[n]$ into k blocks, each consisting of more than one element. We are going to prove that the generating polynomial

$$D_n(x) = \sum_{k \geq 1} D(n,k)x^k \tag{3}$$

has real roots only. We will then use this information to determine the location of the largest coefficient(s) of $D_n(x)$. We also prove that the number of blocks is normally distributed. Finally, we use our methods on r-derangements, and prove the more general result that for any fixed r, the distribution of the number of cycles of r-derangements of length n converges to a normal distribution.

Note that the fact that the two kinds of Stirling numbers behave in the same way under this generalization is not completely expected. Indeed, while $1/e$ of all permutations of length n have no cycles of length 1, and in general, a constant factor of permutations of length n have no cycles of length r or less, the corresponding statement is not true for set partitions.

2

Indeed, almost all partitions of $[n]$ contain a singleton block as we show in Section 3.1. However, as this paper proves, the real zeros property survives.

Finally, we mention that the *vertical* generating functions (minimal block or cycle size is fixed, n varies) of permutations and set partitions have been studied in [8].

2 The Proof of The Real Zeros Property

We start by a recurrence relation satisfied by the numbers $D(n, k)$ of partitions of $[n]$ into k blocks, each block consisting of more than one element. It is straightforward to see that

$$D(n, k) = kD(n - 1, k) + (n - 1)D(n - 1, k - 1). \tag{4}$$

Indeed, the first term of the right-hand side counts partitions of $[n]$ into blocks larger than one in which the element n is in a block larger than two, and the second term of the right-hand side counts those in which n is in a block of size exactly two.

Let $D_n(x) = \sum_{k \geq 1} D(n, k)x^k$. Then (4) yields

$$D_n(x) = x \left(D'_{n-1}(x) + (n - 1)D_{n-2}(x) \right). \tag{5}$$

Note that $D_1(x) = 0$, and $D_n(x) = x$ if $2 \leq n < 4$. So the first non-trivial polynomial $D_n(x)$ occurs when $n = 4$, and then $D_4(x) = 3x^2 + x$. In the next non-trivial case of $n = 5$, we get $D_5 = 10x^2 + x$.

Theorem 1 *Let $n \geq 2$. Then the polynomial $D_n(x)$ has real roots only. All these roots are simple and non-positive. Furthermore, the roots of $D_n(x)$ and $D_{n-1}(x)$ are interlacing in the following sense. If $D_n(x)$ and $D_{n-1}(x)$ are both of degree d, and their roots are, respectively, $0 = x_0 > x_1 > \cdots > x_{d-1}$, and $0 = y_0 > y_1 > \cdots > y_{d-1}$, then*

$$0 > x_1 > y_1 > x_2 > y_2 > \cdots > x_{d-1} > y_{d-1}, \tag{6}$$

while if $D_n(x)$ is of degree $d + 1$ and $D_{n-1}(x)$ is of degree d, and their roots are, respectively, $0 = x_0 > x_1 > \cdots > x_d$, and $0 = y_0 > y_1 > \cdots > y_{d-1}$, then

$$0 > x_1 > y_1 > x_2 > y_2 > \cdots > x_{d-1} > y_{d-1} > x_d. \tag{7}$$

Proof: We prove our statements by induction on n. For $n \leq 4$, the statements are true. Now assume that the statement is true for $n - 1$, and let us prove it for n. Let $0 = y_0 > y_1 > \cdots > y_{d-1}$ be the roots of $D_{n-1}(x)$.

3

First we claim that if $0 > x > y_1$, then $D_{n-1}(x) < 0$, that is, the polynomial S_{n-1} is negative between its two largest roots. Indeed, $S'_{n-1}(0) = S_r(n-1, 1) = 1$, so $D_{n-1}(x)' > 0$ in a neighborhood of 0. This implies that in that neighborhood, $D_{n-1}(x)$ is monotone increasing. As $D_{n-1}(0) = 0$, this implies our claim.

Now consider (5) at $x = y_1$. We claim that at that root, we have both $D'_{n-1}(y_1) < 0$ and $D_{n-2}(y_1) < 0$. The latter is a direct consequence of the previous paragraph and the induction hypothesis. Indeed, the induction hypothesis shows that if z_1 is the largest negative root of D_{n-2}, then $z_1 < y_1 < 0$, and the previous paragraph, applied to D_{n-2} instead of D_{n-1} shows that $D_{n-2}(x) < 0$ if $x \in (z_1, 0)$. So in particular $D_{n-2}(y_1) < 0$. The former follows from Rolle's theorem. Indeed, by Rolle's theorem, between two consecutive roots of D_{n-1}, there has to be a root of D'_{n-1}. As D_{n-1} has simple roots only, say d of them, D'_{n-1} must have $d-1$ simple roots, and therefore, by the pigeon-hole principle, there must be exactly one of them between any two consecutive roots of D_{n-1}. In particular, there is exactly one root of D'_{n-1} between 0 and y_1, so the sign of $D'_{n-1}(y_1)$ is the opposite of the sign of $D'_{n-1}(0) = 1$, that is, it is negative.

So when $x = y_1$, the argument of the previous paragraph shows that the right-hand side of (5) is the product of the negative real number y_1, and the negative real number $D'_{n-1}(y_1) + (n-1)D_{n-2}(y_1)$. Therefore, the left-hand side must be positive, that is, $D_n(y_1) > 0$. As $D_n(x) < 0$ in a neighborhood of 0, this shows that D_n has a root in the interval $(y_1, 0)$.

More generally, we claim that $D_n(x)$ has a root in the interval (y_{i+1}, y_i). For this, it suffices to show that $D_n(y_i)$ and $D_n(y_{i+1})$ have opposite signs. This will follow by (5) if we can show all of the following.

(i) $D'_{n-1}(y_i)$ and $D'_{n-1}(y_{i+1})$ have opposite signs,

(ii) $D_{n-2}(y_i)$ and $D_{n-2}(y_{i+1})$ have opposite signs, and

(iii) $D'_{n-1}(y_i)$ and $D_{n-2}(y_i)$ have equal signs.

Just as before, (i) follows from Rolle's theorem, and (ii) follows from the induction hypothesis. In order to see (iii), note that by Rolle's theorem, D'_{n-1} changes signs exactly i times in $(y_i, 0)$, while by the induction hypothesis, D_{n-2} changes signs $i-1$ times in $(y_i, 0)$. We have seen at the beginning of this proof that in a small neighborhood of 0, D'_{n-1} is positive, while D_{n-2} is negative, so (iii) follows. Therefore, by (5), $D_n(y_i)$ and $D_n(y_{i+1})$ have opposite signs, and so $D_n(x)$ has a root in (y_{i+1}, y_i).

Note that this argument does not *directly* prove that D_n has *exactly* one root in (y_{i+1}, y_i), but it does prove that it has an odd number of roots in each

4

such interval. Indeed, D_n has opposite signs at the endpoints of (y_{i+1}, y_i), it has an odd number of sign changes, and so, an odd number of roots on that interval. As the total number of roots of D_n is at most one larger than that of D_{n-1}, it follows that D_n has indeed exactly one root in each interval (y_{i+1}, y_i).

The above argument completes the proof of the theorem for odd n. When n is even, then D_n is of degree $d+1$, while D_{n-1} is of degree d. In that case, we still have to show that D_n has a root in the interval $(-\infty, y_{d-1})$. However, this follows from the previous paragraph since the last root x_d of D_n must be negative, and cannot be in any of the intervals (y_{i+1}, y_i). $\diamond$

It follows from Theorem 1 that both the sequence $D_4, D_6, D_8, \cdots$, and the sequence $D_5, D_7, D_9, \cdots$ are *Sturm sequences*. The interested reader should consult [15] for the definition and properties of Sturm sequences.

3   Applications of The Real Zeros Property

3.1   Locating peaks

If a polynomial $\sum_{k=1}^n b_k x^k$ with positive coefficients has real roots only, then it is known [3] that the sequence $b_1, b_2, \cdots b_n$ of its coefficients is *strongly log-concave*. That is, for all indices $2 \le j \le n-1$, the inequality

$$b_j^2 \ge b_{j-1} b_{j+1} \frac{j+1}{j} \cdot \frac{n-j+1}{n-j}$$

holds. In other words, the ratio b_{j+1}/b_j is strictly decreasing with j, and therefore there is at most one index j so that $b_{j+1}/b_j = 1$. Thus the sequence $b_1, b_2, \cdots b_n$ has either one peak, or two consecutive peaks.

A useful tool in finding the location of this peak is the following theorem of Darroch.

Theorem 2 *[10] Let $A(x) = \sum_{k=1}^n a_k x^k$ be a polynomial that has real roots only that satisfies $A(1) > 0$. Let m be a peak for the sequence of the coefficients of $A(x)$. Let $\mu = A'(1)/A(1) = \frac{\sum_{k=1}^n k a_k}{\sum_{k=1}^n a_k}$. Then we have*

$$|\mu - m| < 1.$$

Note that in a combinatorial setup, μ is the average value of the statistic counted by the generating polynomial $A(x)$. For instance, if $A(x) = S_n(x)$,

then μ is the average number of blocks in a randomly selected partition of $[n]$.

There is a very extensive list of results on the peak (or two peaks) of the sequence $S(n, 1), S(n, 2), \cdots, S(n, n)$ of Stirling numbers of the second kind. See [9] for a brief history of this topic and the relevant references. In particular [12], if $K(n)$ denotes the index of this peak (or the one that comes first, if there are two of them), then $K(n) \sim n/\log n$. More precisely, let r be the unique positive root of the equation

$$re^r = n. \tag{8}$$

Then, for n sufficiently large, $K(n)$ is one of the two integers that are closest to $e^r - 1$. In view of Theorem 2, one way to approach this problem is by computing the average number of blocks in a randomly selected partition of $[n]$.

Now that we have proved that the generating polynomial $D_n(x) = \sum_{k \geq 1} D(n, k)x^k$ has real roots only, it is natural to ask how much of the long list of results on Stirling numbers can be generalized to the numbers $D(n, k)$. In this paper, we will show a quick way of estimating the average number of blocks in a partition of $[n]$ with no singleton blocks, and so, by Darroch's theorem, the location of the peak(s) in the sequence $D(n, 1), D(n, 2), \cdots, D(n, \lfloor n/2 \rfloor)$. For shortness, let us introduce the notation $D(n) = \sum_k D(n, k)$.

Proposition 1 *For all positive integers $n \geq 2$, the average number $E(n)$ of blocks in a partition of $[n]$ with no singleton blocks is*

$$E(n) = \frac{D(n+1)}{D(n)} - \frac{n(D(n-1))}{D(n)}. \tag{9}$$

Proof: The total number of blocks in all partitions counted by $D(n)$ is clearly $\sum_{k \geq 1} kD(n, k)$. On the other hand,

$$D(n+1) = \sum_{k \geq 1} kD(n, k) + (n-1)D(n-1),$$

since the first term of the right-hand side counts partitions in which the element $n + 1$ is in a block of size three or more, and the second term counts partitions in which the element $n + 1$ is in a block of size two. So $\sum_{k \geq 1} kD(n, k) = D(n+1) - (n-1)D(n-1)$, and the statement follows. $\diamond$

6

Let $B(n)$ denote the number of all partitions of $[n]$. This number is often called a *Bell number*. There are numerous precise results on the asymptotics of the Bell numbers. We will only need the following fact [11].

$$\log B(n) = n \left(\log n - \log \log n + O(1)\right). \tag{10}$$

It is well-known that $B(n) = D(n) + D(n + 1)$. A simple bijective proof is obtained by using the identity map on partitions counted by $D(n)$, and removing the element $n + 1$ and turning each element that shared a block with $n + 1$ into a singleton block in partitions counted by $D(n + 1)$. In particular, this yields that $B(n) > D(n + 1)$, and so $\frac{B(n)}{D(n)} \to \infty$ since $\frac{D(n+1)}{D(n)} \to \infty$. (To see the latter, note that for any h, there exists an N so that if $n > N$, then almost all partitions counted by $D(n)$ have more than h blocks.) Therefore, $D(n + 1) \sim B(n)$, and (9) implies

$$\frac{B(n)}{B(n-1)} - E(n) = \frac{nB(n-2)}{B(n-1)} \sim \log n.$$

An argument that is very similar to (but even simpler than) the proof of Proposition 1 shows that the average number of blocks in an *unrestricted* partition of $[n]$ is $\frac{B(n+1)}{B(n)} - 1$. Using Theorem 2, this leads to the following. Recall that $K(n)$ is the location of the (leftmost) peak of the sequence $S(n, 1), S(n, 2), \cdots, S(n, n)$.

Theorem 3 *Let n be a fixed positive integer, and let $k(n)$ be the index for which $D(n, k(n))$ is maximal, or if there are two such indices, then the smaller one. Then $K(n-1) - k(n) \sim \log n$.*

Note that this result is in line with what we would intuitively expect, since a randomly selected partition of $[n]$ has, on average, about $\frac{n}{\log n}$ blocks, about $\log n$ of which are singleton blocks, as can be seen by computing the probability that $\{1\}$ is a singleton block by (10), then using the linearity of expectation.

3.2 Proving Normal Behavior

Another frequent application of the real zeros property is proving that the coefficients of the polynomial at hand are normally distributed. Our tool in doing so will be a theorem of Ed Bender, proved in [1]. The reader may also want to consult the survey paper of Andrzej Rucinski for this and other methods to prove normality in combinatorics. Let X_n be a random

variable, and let $a_n(k)$ be a triangular array of non-negative real numbers, $n = 1, 2, \cdots$, and $1 \leq k \leq m(n)$ so that

$$P(X_n = k) = p_n(k) = \frac{a_n(k)}{\sum_{i=1}^{m(n)} a_n(i)}.$$

Set $g_n(x) = \sum_{k=1}^{m(n)} p_n(k) x^k$.

We need to introduce some notation for transforms of the random variable Z. Let $\bar{Z} = Z - E(Z)$, let $\tilde{Z} = \bar{Z}/\sqrt{\mathrm{Var}(Z)}$, and let $Z_n \to N(0,1)$ mean that Z_n converges in distribution to the standard normal variable.

Now we are ready to state Bender's theorem.

Theorem 4 *[1] Let X_n and $g_n(x)$ be as above. If $g_n(x)$ has real roots only, and*

$$\sigma_n = \sqrt{\mathrm{Var}(X_n)} \to \infty,$$

then $\tilde{X}_n \to N(0,1)$.

See [6] for related results.

We are going to use this theorem in the special case when X_n is the number of blocks in a randomly selected partition of $[n]$ with no singleton blocks, that is, $a_n(k) = D(n,k)$. Then $p_n(k) = D(n,k)/D(n)$ for all k, so Theorem 1 implies that $g_n(x) = \frac{1}{D(n)} \sum_{k \geq 1} D(n,k) x^k$ has real roots only.

There remains the task of showing that $\sigma_n = \sqrt{\mathrm{Var}(X_n)} \to \infty$. We will accomplish that following the lead of the classic paper of Harper [12], in which he proved that the number of blocks of *unrestricted* partitions of $[n]$ is normally distributed. We will have a little bit more details to handle, but the crucial ideas of Harper's proof survive. The first step is the following proposition.

Proposition 2 *For all positive integers $n \geq 2$, we have*

$$\sum_{k=1}^{\lfloor n/2 \rfloor} D(n,k) k^2 = D(n+2) - (2n+1)D(n) - nD(n-1) - n(n-1)D(n-2).$$

Proof: Take a partition of $[n]$ into k blocks, none of which are singletons. Insert the entries $n+1$ and $n+2$ in any of these blocks in k^2 ways. Summing over k, this shows that there are $\sum_{k=1}^{\lfloor n/2 \rfloor} D(n,k) k^2$ partitions of $[n+2]$ into blocks, none of which are singletons, so that $n+1$ and $n+2$ are in blocks that contain at least two elements that are at most n. How many other

8

partitions of $[n + 2]$ are there without singleton blocks? There are $D(n)$ such partitions in which $\{n + 1, n + 2\}$ is a block. By the sieve formula, there are $2nD(n) - n(n-1)D(n-2)$ such partitions in which there is either a block of the form $\{i, n+1\}$ for some $i \leq n$, or a block of the form $\{j, n+2\}$, or both. Finally, there are $nD(n-1)$ such partitions in which $\{i, n+1, n+2\}$ is a block for some $i \leq n$. $\diamond$

Lemma 1 *Let X_n be the number of blocks in a randomly selected partition of $[n]$ with no singleton blocks. Then $Var(X_n) = \sigma_n \to \infty$.*

Proof: We use the identity $\sigma_n^2 = E(X_n^2) - E(X_n)^2$. Note that $E(X_n^2) = \frac{\sum_{k=1}^{\lfloor n/2 \rfloor} D(n,k)k^2}{D(n)}$, and the numerator of this fraction was computed in Proposition 2. Furthermore, $E(X_n) = E(n)$ was computed in (9). That is,

$$E(X_n^2) = \frac{D(n+2)}{D(n)} - \frac{nD(n-1)}{D(n)} - \frac{n(n-1)D(n-2)}{D(n)} - (2n+1),$$

and

$$E(X_n)^2 = \frac{D(n+1)^2}{D(n)^2} + n^2 \frac{D(n-1)^2}{D(n)^2} - 2n \frac{D(n-1)D(n+1)}{D(n)^2}.$$

Therefore,

$$\sigma_n^2 = \left(\frac{D(n+2)}{D(n)} - \frac{D(n+1)^2}{D(n)^2} \right) +$$

$$+ 2n \left(\frac{D(n-1)D(n+1)}{D(n)^2} - 1 \right)$$

$$- \frac{n^2 D(n-1)^2 + nD(n)D(n-1) + n(n-1)D(n)D(n-2)}{D(n)^2} - 1.$$

We will now consider the first three terms of the right-hand side one by one. It is proved in Lemma 2 of [12] that $\left(\frac{B(n+2)}{B(n)} - \frac{B(n+1)^2}{B(n)^2} \right) \to \infty$, in fact, it is proved there that

$$\left(\frac{B(n+2)}{B(n)} - \frac{B(n+1)^2}{B(n)^2} \right) = \frac{n}{r(r+1)} + o(1),$$

where r is defined as in (8). As we mentioned in the paragraph preceding (8), we know that $e^r \sim n/\log n$, so $r \sim \log n - \log \log n$, and so

9

$\left(\frac{B(n+2)}{B(n)} - \frac{B(n+1)^2}{B(n)^2}\right) \sim n/\log^2 n$. As we know that $B(n) \sim D(n+1)$, it follows that $\left(\frac{D(n+2)}{D(n)} - \frac{D(n+1)^2}{D(n)^2}\right) \sim n/\log^2 n$.

It is proved in [7] that the sequence $D(2), D(3), \cdots$ is log-convex, that is, $D(n-1)D(n+1) \geq D(n)^2$ for all $n \geq 3$. Therefore, the second term is non-negative.

Finally, it is routine to prove, using the fact that $\frac{D(n)}{D(n-1)} \sim \frac{B(n-1)}{B(n-2)}$ is roughly $n/\log n$, that the third term is of order $c\log^2 n$. Therefore, $\sigma_n \to \infty$ as claimed. $\diamond$

We have seen that both conditions of Theorem 4 hold. Therefore, we have proved the following theorem.

Theorem 5 *Let X_n denote the number of blocks in a randomly selected partition of $[n]$ with no singleton blocks. Then $\tilde{X}_n \to N(0,1)$.*

3.3 The normality of the number of cycles of r-derangements

Let r be a fixed non-negative integer, and let Y_n be the number of cycles in a randomly selected r-derangement of length n. Recall from the introduction that an r-derangement is a permutation in which each cycle is longer than r. Theorem 4 and a recent result of present author provide a very quick way of proving that $\tilde{Y}_n \to N[0,1]$.

Recall from the introduction that $d_r(n,k)$ is the number r-derangements of length n that have k cycles. Then it is known [5] that the generating polynomial $d_{n,r}(x) = \sum_{k \geq 1} d_r(n,k)x^k$ has real roots only. So one of the two conditions of Theorem 4 holds for the variables Y_n. We need to show that the other one holds as well, that is, that $\sqrt{\text{Var}(Y_n)} \to \infty$. We need the following tools.

Proposition 3 *[1] Let X_n and $g_n(x)$ be defined as in Theorem 4. If $g_n(x)$ has real roots only, then*

$$Var(X_n) = \sum_{j=1}^{m(n)} \frac{\lambda_j}{(1 + \lambda_j)^2},$$

where $-\lambda_1, -\lambda_2, \cdots$ are the roots of $g_n(x)$.

Proof: Define independent random variables $X_{n,1}, X_{n,2}, \cdots, X_{n,m(n)}$ so that they are all 0-1 variables, and $P(X_{n,j} = 0) = \frac{\lambda_j}{1+\lambda_j}$, and $P(X_{n,j} = 1) =$

$\frac{1}{1+\lambda_j}$. It is then straightforward to verify that $\sum_{j=1}^{m(n)} X_{n,j} = X_n$ by checking that $g(n)$ is the product of the probability generating functions of the $X_{n,j}$. Therefore, $\mathrm{Var}(X_n) = \sum_{j=1}^{m(n)} \mathrm{Var}(X_{n,j}) = \sum_{j=1}^{m(n)} \frac{\lambda_j}{(1+\lambda_j)^2}$ as claimed. $\diamond$

In other words, if we know the roots of the generating function $g_n(x)$, then we can compute of $\mathrm{Var}(X_n)$. Note that we do not need to compute $\mathrm{Var}(X_n)$; we only need to show that $\mathrm{Var}(X_n) \to \infty$. To that end, we do not necessarily need to know *all roots* λ_j; sometimes it is enough to know just some of them. The following theorem provides sufficient information about the roots in the special case when $X_n = Y_n$, and so $g_n(x)$ is a constant multiple of $d_{n,r}(x)$.

Theorem 6 *[4] For every negative integer $-t$, and every $\epsilon > 0$, there exists a positive integer N so that if $n > N$, then $d_{n,r}(x)$ has a root x_t satisfying $|-t - x_t| < \epsilon$.*

Therefore, for any positive integer t, and any $\epsilon > 0$, we can find an N so that if $n > N$, then

$$\mathrm{Var}(Y_n) \geq \sum_{k=1}^{t} \frac{k}{(k+1)^2} - \epsilon,$$

and therefore, $\mathrm{Var}(Y_n) \to \infty$. So we have verified that both conditions of Theorem 4 hold, and therefore, we have proved the following theorem.

Theorem 7 *Let r be a fixed non-negative integer, and let Y_n be the number of cycles in a randomly selected r-derangement of length n. Then $\tilde{Y}_n \to N[0,1]$.*

4   Further Directions

It is natural to ask whether Theorem 1 can be generalized to partitions with all blocks larger than r, where r is a given positive integer. That would parallel the result (2) of Brenti [5] on permutations.

A consequence of the fact that the polynomials $D_n(x)$ have real zeros is that for any fixed n, the sequence $D(n,1), D(n,2), \cdots, D(n, \lfloor n/r \rfloor)$ is log-concave. In [14], Bruce Sagan provides a proof for the special case of $r = 0$, that is, that of the classic Stirling numbers of the second kind. However, his injection proving that result does not preserve the no-singleton-block property. Now that we know that the statement is true, it is natural to ask

11

for an injective proof. Similarly, as we know that $d_{n,r}(x)$ has real roots only (see (1)), we can ask for a combinatorial proof for the fact that the sequence $d_r(n,1), d_r(n,2), \cdots, d_r(n, \lfloor n/r \rfloor)$ is log-concave.

Acknowledgment

References

[1] E. A. Bender, Central and Local Limit Theorems Applied to Asymptotic Enumeration, *Journal of Combinatorial Theory, Ser. A*, **15** (1973), 91–111.

[2] M. Bóna, **A Walk Through Combinatorics**, second edition, *World Scientific,* 2006.

[3] M. Bóna, **Introduction to Enumerative Combinatorics**, *McGraw-Hill*, 2007.

[4] M. Bóna, *On a balanced property of derangements,* Electronic Journal of Combinatorics, **13** (2006), R102.

[5] F. Brenti, Permutation enumeration, symmetric functions, and unimodality. *Pacific J. Mathematics*, **157** (1993), no. 1, 1–28.

[6] E. R. Canfield, Central and local limit theorems for coefficients of polynomials of binomial type, *Journal of Combinatorial Theory, Ser. A*, **23** (1977), 275–290.

[7] E. R. Canfield, E. A. Bender, Log-concavity and related properties of the cycle index polynomials. *J. Combin. Theory Ser. A* **74** (1996), no. 1, 57–70.

[8] E. R. Canfield, Asymptotic Normality in Binomial Type Enumeration, *Doctoral Dissertation*, University of California San Diego, 1975.

[9] E. R. Canfield, C. Pomerance, On the problem of uniqueness for the maximum Stirling numbers of the second kind, *Integers* **2** (2002), #A01.

[10] J. N. Darroch, On the distribution of the number of successes in independent trials. *Ann. Math. Statist.* **35** (1964), 1317–1321.

[11] N. G. de Bruijn, **Asymptotic Methods in Analysis**, *Dover*, pp. 102-109, 1981.

[12] L. H. Harper, Stirling behavior is asymptotically normal, *Ann. Math. Statist.* **38** (1967), 410–414.

[13] A. Rucinski, Proving Normality in Combinatorics, in *Random Graphs*, Volume 2, Wiley Interscience, 1992, 215–231.

[14] B. Sagan, Inductive and injective proofs of log concavity results, *Discrete Math.*, **68** (1998), no. 2-3, 281–292.

[15] H. Wilf, **Mathematics for the physical sciences**, Reprinting of the 1962 original. Dover Books in Advanced Mathematics, *Dover Publications, Inc*, New York, 1978.

[16] H. Wilf, **Generatingfunctionology**, second edition. *Academic Press*, 1994.